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We study about approximation method of the Hawking radiation in exotic black hole backgrounds. To consider it, we investigate models which have peculiar properties in black hole thermodynamics (monopole black hole in SO(3) Einstein-Yang-Mills-Higgs system and dilatoic black hole in Einstein-Maxwell-dilaton system). For simplicity, we consider a massless scalar field which does not couple to matter fields. In this case, we can well approximate the Hawking radiaition with 'black body' type radiation described below. We also discuss its validity.

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The Hawking radiaition [1] around black holes has been discussed for many years concerning various aspects, e.g., as γ-ray sources of the early universe [2] or vacuum polarization of charged black holes [3], etc. But as for black holes with non-Abelian hair [4–10], it has not been much investigated because many of them are only obtained numerically that it takes much works compared with ones of analytically obtained. But we need to investigate for many reasons. Particularly, a monopole black hole which was found in SO(3) Einstein-Yang-Mill-Higgs (EYMH) system [11–14] is important because it is one of the counterexample of the black hole no hair conjecture [15]. Moreover, if we consider the evaporation process of the Reissner-Nortström (RN) black hole, it may become a monopole black hole and the regular gravitating monopole is the candidate of the remnant of the Hawking radiation in this system.

From such view points, we consider approximation method of the Hawking radiation which reduce our labor and discuss its validity in this paper. Throughout this paper we use units $c = \hbar = 1$. Notations and definitions as such as Christoffel symbols and curvature follow Misner-Thorne-Wheeler [16].

We consider two models in which black hole solutions have peculiar properties in black hole thermodynamics.

(I) The SO(3) EYMH model as

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} + L_m\right], \tag{1}$$

where $\kappa^2 \equiv 8\pi G$ with G being Newton's gravitational constant. L_m is the Lagrangian density of the matter fields which are written as

$$L_m = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2}(D_\mu\Phi^a)(D^\mu\Phi^a) - \frac{\lambda}{4}(\Phi^a\Phi^a - v^2)^2 \ . \tag{2}$$

$F^a_{\mu\nu}$ is the field strength of the SU(2) YM field and expressed by its potential A^a_μ as

$$F^a_{\mu\nu} = \partial_\mu A^a_\mu - \partial_\nu A^a_\nu + e\epsilon^{abc}A^b_\mu A^c_\nu, \tag{3}$$

with the gauge coupling constant e. Φ^a is the real triplet Higgs field and D_μ is the covariant derivative:

$$D_\mu\Phi^a = \partial_\mu\Phi^a + e\epsilon^{abc}A^b_\mu\Phi^c. \tag{4}$$

The theoretical parameters v and λ are the vacuum expectation value and the self-coupling constant of the Higgs field, respectively.

(II) The Einstein-Maxwell-dilaton (EMD) model as

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R - 2(\nabla\phi)^2 - e^{-2\alpha\phi}F^2\right], \tag{5}$$

where ϕ and F are a dilaton field and U(1) gauge field, respectively.

For black hole solutions, we assume that a space-time is static and spherically symmetric, in which case the metric is written as

$$ds^2 = -f(r)e^{-2\delta(r)}dt^2 + f(r)^{-1}dr^2 + r^2d\Omega^2, \tag{6}$$

where $f(r) = 1-2Gm(r)/r$. We consider solutions which have regular horizon and are asymptotically flat. Around them, we consider a neutral and massless scalar field which does not couple to the matter fields. This is described by the Klein-Gordon equation as

$$\Phi_{,\mu}{}^{;\mu} = 0. \tag{7}$$

The energy emission rate of Hawking radiation is given by

$$\frac{dM}{dt} = -\frac{1}{2\pi}\sum_{l=0}^{\infty}(2l+1)\int_0^\infty \frac{\Gamma(\omega)\omega}{e^{\omega/T_H}-1}d\omega, \tag{8}$$

where l and $\Gamma(\omega)$ are the angular momentum and the transmission probability in a scattering problem for the scalar field Φ. ω and T_H are the energy of the particle and the Hawking temperature respectively. We define as $\Xi \equiv -dM/dt$.

The Klein-Gordon equation (7) can be made separable, and we should only solve the radial equation

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$$\frac{d^2\chi}{dr^{*2}} + \chi\left[\omega^2 - V^2\right] = 0, \tag{9}$$

where

$$V^2 \equiv \frac{f}{re^{2\delta}}\left\{\frac{l(l+1)}{r} - f\delta' + \frac{2(m - m'r)}{r^2}\right\}, \tag{10}$$

$$\frac{dr}{dr^*} \equiv fe^{-\delta}, \tag{11}$$

where $'$ denote d/dr and χ is only the function of r. The transmission probability Γ can be calculated by solving radial equation numerically under the boundary condition

$$\chi \to Ae^{-i\omega r^*} + Be^{i\omega r^*} \quad (r^* \to \infty), \tag{12}$$

$$\chi \to e^{-i\omega r^*} \quad (r^* \to -\infty), \tag{13}$$

where Γ is given as $1/|A|^2$. As we see the dominant contribution for the Hawking radiation is $l = 0$, because the contribution of the higher modes are suppressed by the centrifugal barrier. In what follows, we ignore the contributions from $l \geq 2$. Since the systematic error caused by it is below 1%, this does not affect the discussion below. The evaporation process of the monopole black hole and the dilatonic black holes are discussed in [17] and in [18], respectively.

Next, we consider an approximation method of the Hawking radiation. Naively speaking, the Hawking radiation is a blackbody radiation of the temperature T_H. Then we may think that T_H mainly contributes to the Hawking radiation and decides the evaporation rate of a black hole. However we have another factor, i.e., transmission amplitude Γ. But Γ depends on the energy of the particle, so in general we must solve a radial equation numerically and integrate (8) again numerically. Moreover, we have to solve the background of the metric numerically for non-Abelian black holes. It would be rather a troublesome task.

But since the emission is of a blackbody nature, we may be able to estimate it by Stefan's law [19] as $-dM/dt \sim \pi\sigma R_{eff}^2 T_H^4$. $\sigma = \pi^2/15$ is the Stefan-Boltzmann constant. The unknown is the effective radius R_{eff}. It was suggested that for Schwarzschild black hole, R_{eff} is given by the photon orbit with some energy E and angular momentum L. Although such a formula provides a good approximation, one may wonder whether it is still valid for exotic black holes, which have envelope outside of event horizon.

If we consider the interactions between matter fields and the scalar field, it would change the result completely. Here, we ignore them for simplicity. Starting with metric (6) we find the maximum point of the effective potential V_{eff} defined by

$$V_{eff} \equiv \frac{L^2}{2r^2}\left(1 - \frac{2Gm}{r}\right), \tag{14}$$

and then the effective radius as

$$R_{eff} \equiv \frac{L}{E} = \left(\frac{R^3}{R - 2m}\right)^{1/2} e^{\delta}, \tag{15}$$

where R is the radial cordinate where V_{eff} takes its maximum value. R_{eff} is determined by the critical value of the "impact parameter" below which any photon sent toward the black hole can not escape. Now that we discuss the Hawking radiation of a massless field. We define $\Xi_{BB} \equiv \pi^3 R_{eff}^2 T_H^4/30$ as a blackbody approximation.

First, we examine how well we can approximate by this method in monopole and RN black holes. We show the typical relation between the Hawking temperature T_H and the ratio Ξ/Ξ_{BB} in Fig. 1. We choose as $\lambda/e^2 = 0.1$, $v/M_{pl} = 0.05$. The arrow shown in this diagram shows the direction of the evaporating process. We find that this quickly change below the point A which corresponds to the change of the sign of the specific heat. We restrict our calculation below $Q/Q_{max} = 0.99$ where Q_{max} shows the maximum charge. Note that above parameters are not particular ones. Actually, we obtained similar results for other parameters. We can summerize the results as follows.

(i) The approximation becomes wrong near the extreme limit.

(ii) The temperature does not necessarily decide validity of this approximation. For example, if we have two solutions for the same temperature as in Fig. 1, near extreme solution have worse result.

We want to know what condition should be imposed for good approximation. To clarify this, we consider the EMD model where exact black hole solutions are obtained [20]. One of the reason to choose this model is α dependence of the solutions. Particularly, α changes the temperature and the shapes of the effective potential near the extreme solution. Roughly speaking, above method is of WKB nature, we may think its validity by two factors, steepness of the effective potential and the temperature which determines the frequency of the particle which contributes to the emission rate.

We show shapes of the potentials for (a) $\alpha = 0$ (b) $\alpha = 1$ (c) $\alpha = 2$ in terms of r^* in Fig. 2 [21]. The parameters Q/Q_{max} are chosen to be 0, 0.84, 0.89, 0.94, 0.99 in (a) and (b) and to be 0, 0.81, 0.86, 0.9, 0.95 in (c). As we increase α, the difference from the Schwarzschild black hole increases near the extreme limit, i.e., the shapes of the potential becomes steep and the temperature becomes high. So its a competition between steepness of the potential which disturbs approximating and the increasing temperature which justifies the approximation.

We show the relation between Q/Q_{max} and Ξ/Ξ_{BB} for $\alpha = 0$, 0.5, 1.0, 1.5, 2.0 in Fig. 3. For $Q/Q_{max} \lesssim 0.8$, the difference is about 6%, and it seems to be rather universal. But for $Q/Q_{max} \gtrsim 0.8$, results depends on α remarkably. It may seem curious because the value Ξ/Ξ_{BB} near the extreme solution does not monotonically increase by the increase of α (The order for 'good' approximation is $\alpha = 2.0$, 1.5, 0.0, 1.0, 0.5.), but we can interpret it easily.

We show the relation between T_H and Ξ/Ξ_{BB} in Fig. 4 for the same parameters in Fig. 3. The arrows show the direction to the evaporation process. If we compare the Ξ/Ξ_{BB} for fixed T_H, it monotonically depends on the 'steepness' of the potential which can be evaluated using R_{eff} because it decreases when V_{eff} becomes steep. We should consider both T_H and R_{eff} to evaluate this approximation.

We show $T_H R_{eff}$ in terms of Q/Q_{max} in Fig. 5. One can see that $T_H R_{eff}$ roughly resembles to the Ξ/Ξ_{BB}. (The exceptional case is $\alpha = 1$ which shows that steepness of the potential may become more important to evaluate this method than the increasing of the temperature.) So this may indicate that the validity of 'black body' approximation can be roughly evaluated by two factors, T_H and R_{eff}. It is surprising that this crude approximation provide such good results and it may provide the effective way to evaluate the Hawking radiation.

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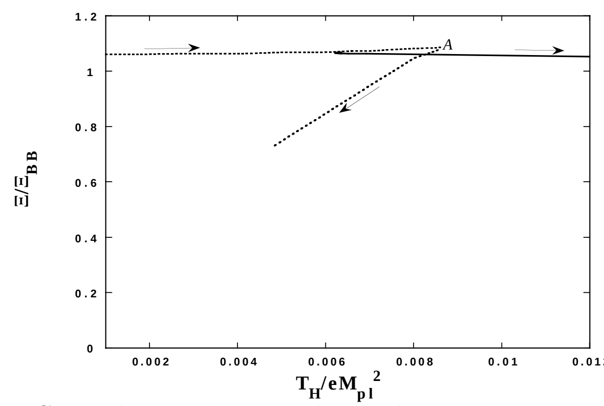

FIG. 1. The ratio between the Hawking radiation and the black body approximation Ξ/Ξ_{BB} in terms of the Hawking temperature T_H/eM_{pl}^2 for $\lambda/e^2 = 0.1$, $v/M_{pl} = 0.05$. We show the RN black hole as dotted lines and the monopole black hole as solid lines. The point A correspond to the change of the sign of the specific heat. We can find that though Ξ_{BB} differs from Ξ about 6%, its difference seems rather universal except below the point A.

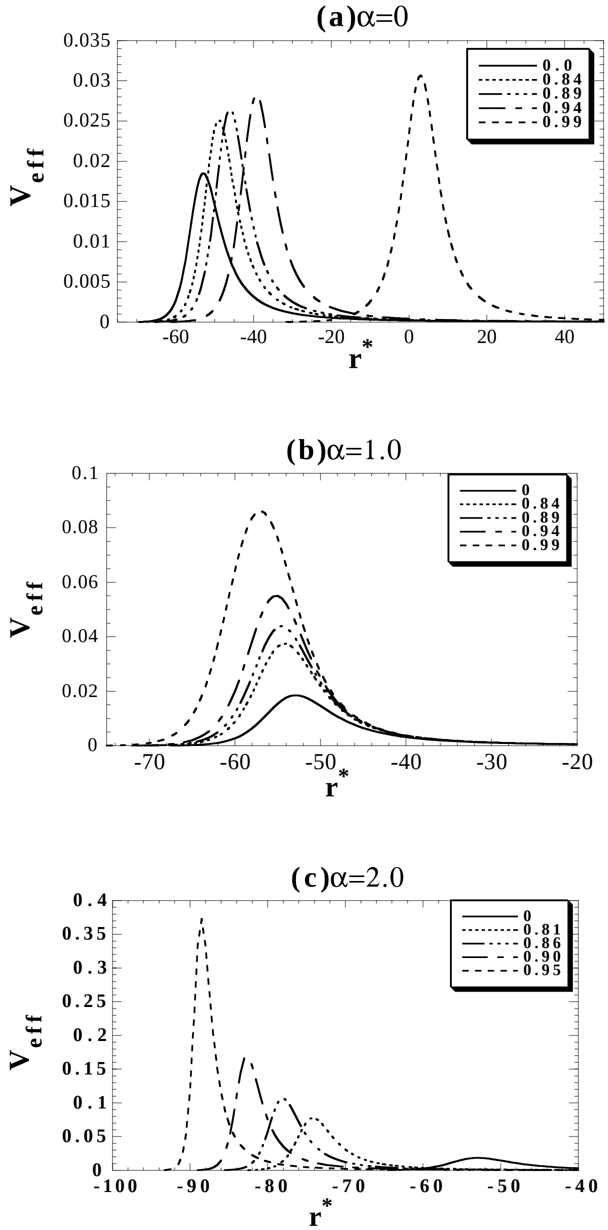

FIG. 2. Shapes of the potential V_{eff} in terms of r^* for (a) $\alpha = 0$ (b) $\alpha = 1$ (c) $\alpha = 2$. For the parameter Q/Q_{max}, we choose to be 0, 0.84, 0.89, 0.94, 0.99 in (a) and (b) and to be 0, 0.81, 0.86, 0.9, 0.95 in (c). Note that for large α, the shapes of the potential extremely change near the extreme limit.

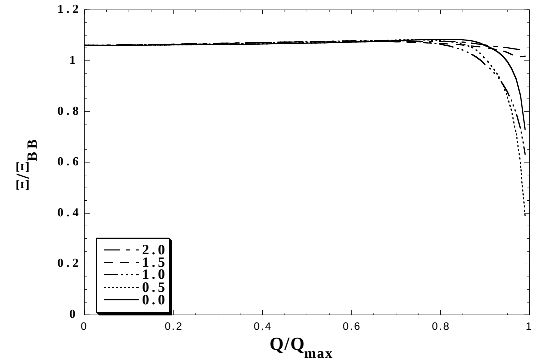

FIG. 3. The ratio between the Hawking radiation and the black body approximation Ξ/Ξ_{BB} in terms of Q/Q_{max} for dilatonic black holes for $\alpha = 0$, 0.5, 1, 1.5, 2 which shows that near the extremal limit, Ξ/Ξ_{BB} change rapidly.

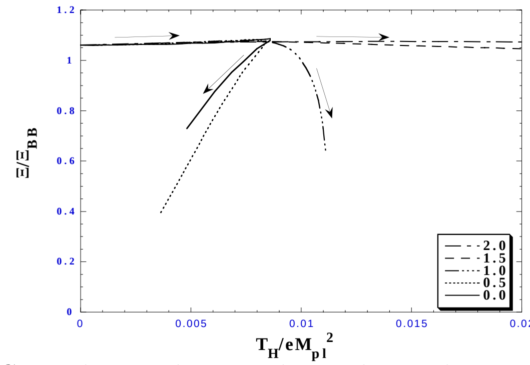

FIG. 4. The ratio between the Hawking radiation and the black body approximation Ξ/Ξ_{BB} in terms of Hawking temperature T_H/eM_{pl}^2 for the same solutions in Fig. 3. The arrows show the decrease of the gravitational mass if the electric charge is fixed. This may suggests that the approximation becomes invalid for the decrease of T_H or approaching the extreme black hole.

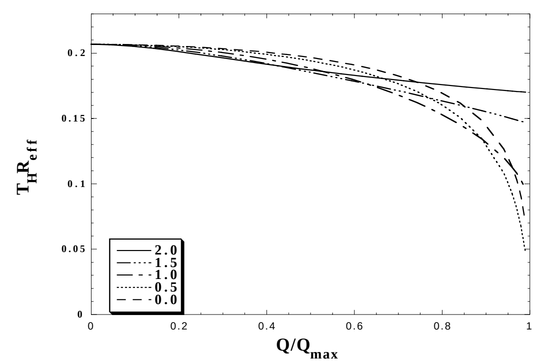

FIG. 5. $T_H R_{eff}$ in terms of Q/Q_{max} which shows that this can be the indicator to evaluate the validity of the approximation.